TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 282, Number 2, April 1984

INTERMEDIATE NORMALIZING EXTENSIONS[1]

BY

A. G. HEINICKE AND J. C. ROBSON

ABSTRACT. Relationships between the prime ideals of a ring R and of a normalizing extension S have been studied by several authors recently. In this work, most of these known results are extended to give relationships between the prime ideals of R and of T where T is a ring with $R \subset T \subset S$, and S is a normalizing extension of R: such rings T are called *intermediate normalizing extensions* of R.

One result ("Cutting Down") asserts that for any prime ideal J of T, $J \cap R$ is the intersection of a finite set of prime ideals P_i of R, uniquely defined by J, whose corresponding factor rings R/P_i are mutually isomorphic. The minimal members of the family of P_i's are the primes of R minimal over $J \cap R$, and an "incomparability" theorem is proved which shows that no two comparable primes of T can have their intersections with R share a common minimal prime. Other results include versions of the "lying over" and "going up" theorems, proofs that chain conditions such as right Goldie or right Noetherian pass between T/J and each of the rings R/P_i, and a demonstration that the "additivity principle" holds.

1. Introduction. Suppose that R is a subring of S, sharing the same identity element, and that S is finitely generated as an R-module by elements $a_1, \ldots, a_n$ with $a_iR = Ra_i$. Then S is called a *normalizing extension* of R. The relationship between the prime ideals of these two rings has been extensively studied, both by the authors **[5, 6, 7]** and by Lorenz **[11]**, Passman **[13]**, Lanski **[10]** and others **[2, 3, 14, 17]**. This relationship is very similar to that of Krull, and of Cohen and Seidenberg, for integral extensions of a commutative ring. Indeed, it is known **[11**, 1.3**]** that, in a sense, S is integral over R.

This suggests that a similar relationship could exist between the prime ideals of R and those of any ring T with $R \subset T \subset S$, such a ring T being termed an *intermediate normalizing extension* of R. In fact there are already results about this relationship when the a_i centralize R **[15, 16]** and in some other special cases **[9]**.

In this paper a description, almost as complete as that for normalizing extensions, is given of the relationship between the prime ideals of R and any intermediate normalizing extension T. It is shown that, associated with any prime ideal J of T, there is a set $\{P_1, \ldots, P_k\}$ of at most n (the number of generators of S) prime ideals of R, uniquely defined by J; these are said to be *connected* to J. Moreover, as the "cutting down" theorem asserts, $\bigcap P_i = J \cap R$, and the prime rings R/P_i are all isomorphic. It follows, of course, that $J \cap R$ is semiprime and has finitely many minimal primes.

Received by the editors March 15, 1983.

1980 *Mathematics Subject Classification*. Primary 16A56, 16A66; Secondary 16A26, 16A34, 16A55.

[1]Research was supported in part by a grant from the Natural Sciences and Engineering Research Council of Canada, Grant A-7338.

An "incomparability" theorem is also proved. This says that no two comparable primes of T can have the same intersection with R; in fact, their intersections cannot share a common minimal prime. This is obtained as a corollary to the "essentiality" theorem which proves that any nonzero ideal of T/J is actually essential as an R-subbimodule of T/J. There are appropriate versions of "lying over" and "going up" theorems, the latter being new even when S is a centralizing (liberal) extension of R. Also it is shown that chain conditions such as right Goldie or right Noetherian pass between T/J and each of the rings R/P_i, and that the "additivity principle" holds.

In obtaining these results, the paper depends upon the ideas and results of the authors' earlier papers [**5**, **7**] concerning the relationship between R and S, especially when S is prime. In that case [**7**] found and used a particular collection of idempotent elements $\{f_1,\dots,f_m\}$ in the Martindale quotient ring of S. These are used in similar fashion here to give, for each i, another ring T_i which is linked with T through a Morita context and with $T_i \supset f_iR \simeq R/P_i$. The general strategy is to reduce questions concerning T to similar questions concerning T_i. There is a snag in dealing with the questions for T_i, since T_i is not quite an intermediate normalizing extension of f_iR. Nevertheless, it is near enough for the techniques of [**8**] to be usable; and the results about T_i make up an appendix to this paper.

The description of T_i comes in §2 together with an analysis of the nature of T/J as a (T, R)-bimodule. This culminates in the "cutting down" theorem. Similarly, §3 obtains "essentiality" and "incomparability" by conducting an analysis of T/J as an (R, R)-bimodule. "Lying over" and "going up" come in §4, together with some similar results for prime ideals of T and S. Easy examples show that, if I is a prime ideal of S, $I \cap T$ need not be semiprime. But, as §5 shows, $T/I \cap T$ has at most n minimal primes, and its prime radical is nilpotent of index at most n. This gives connections between the prime radicals of R, T and S. Similar results for primitive ideals and the Jacobson radical also come in §§4 and 5. Finally, chain conditions are discussed in §6.

It is surely clear from above that the papers [**5**, **7**] are prerequisites for this paper. The relevant constructions and theorems will be described and quoted as appropriate.

2. Standard settings and cutting down. Suppose that S is a normalizing extension of R, that T is an intermediate normalizing extension with $R \subset T \subset S$, and that J is a prime ideal of T. In studying J and T/J, one can usually reduce problems to the case in which

(i) S is a prime ring, and

(ii) $B \cap T \not\subset J$ whenever $0 \neq B \lhd S$;

this case being described as a *standard setting* for J. In this section the reduction is described, and then the structure of T/J as a (T, R)-bimodule is deduced from the facts [7] concerning ${}_SS_R$. This leads to the Cutting Down Theorem.

LEMMA 2.1. *Let $R \subset T \subset S$ with S a prime normalizing extension of R, and let J be a prime ideal of T. This is a standard setting if and only if J is not essential in ${}_RT_R$ (i.e., $J \notin \mathfrak{S}({}_RT_R)$).*

PROOF. Corollary 2.25 of [7] shows that if ${}_RW'_R \subset {}_RW_R \subset {}_RS_R$ then $W' \in \mathcal{E}({}_RW_R)$ if and only if $W \cap B \subset W'$ for some $0 \neq B \lhd S$. Take $W = T$ and $W' = J$. $\square$

PROPOSITION 2.2. *Let $R \subset T \subset S$ with S a normalizing extension of R, and let J be a prime ideal of T. There is an ideal I of S with $I \cap T \subset J$ such that, with identifications of subrings,*

$$R/I \cap R \rightarrowtail T/I \cap T \rightarrowtail S/I$$

gives a standard setting for $J/I \cap T$.

PROOF. By Zorn's lemma, there is an ideal I of S maximal with respect to $I \cap T \subset J$. The inclusions above give $T/I \cap T$ as an intermediate extension of $R/I \cap R$, and give a standard setting for $J/I \cap T$, the verifications being routine. $\square$

NOTATION 2.3. For the study of $R/J \cap R$ and T/J, 2.2 makes clear that one may assume that the setting is standard for J (and then I is zero and S is prime). Notation is fixed as such until specified otherwise (in the final few results).

Use will be made of results from [7] concerning the structure of S, and a brief survey of the relevant information follows.

First, S embeds in $Q = Q(S)$, the Martindale right quotient ring of S. In Q there is a family $f_1, \ldots, f_m$ of orthogonal idempotents which sum to 1 and which centralize R. The set $\{P_i \mid 1 \leqslant i \leqslant m\}$, with $P_i = \operatorname{ann}_R(f_i)$ is a set of m (different) prime ideals of R: these are the primes of R *connected* to the zero ideal of S. Although proper inclusions may occur among the ideals P_i, $R/P_i \cong R/P_j$ for all i and j.

Since $0 = \bigcap_1^m P_i$, R is semiprime and embeds into the subring R^* of Q, where $R^* = \sum_1^{\oplus m} f_i R$, a direct product of the prime rings $f_i R$ ($\cong R/P_i$). Also Q decomposes as $\sum_{i,j=1}^{\oplus m} f_i Q f_j$, this being a direct sum of (R, R) and of (R^*, R^*)-bimodules. For any $W \lhd {}_RQ_R$, set $W_{ij} = W \cap f_i W f_j = W \cap f_i Q f_j$. Then $\sum_{i,j=1}^m W_{ij}$ is denoted by W°, this being termed the *interior* of W. From [7, 4.5 and 4.6], W° is the largest R^*-subbimodule of Q contained in W, and ${}_RW_R^\circ$ is essential in ${}_RW_R$.

Corollary 2.25 of [7] (cited in the proof of 2.1) can be applied to $T^\circ \subset T$, showing that there is a nonzero ideal I_0 of S such that $I_0 \cap T \subset T^\circ$. *This notation will now be fixed*. Since the setting is standard for J, then $I_0 \cap T \not\subset J$.

For each idempotent f_i, the subsets $T_i = T_{ii} + f_i R = (T \cap f_i Q f_i) + f_i R$ and $S_i = S_{ii} + f_i R$ are subrings of Q. There is a chain

$$f_i R \subset T_i \subset S_i \subset f_i S f_i \subset f_i Q f_i$$

and $f_i S f_i$ is a normalizing $f_i R$-bimodule, generated by the $f_i a_k f_i$ where $a_1, \ldots, a_n$ are R-normalizing generators of S. Furthermore, each S_i is a prime ring with $Q(S_i) \simeq f_i Q f_i$ [7, 2.18]. This chain, with T_i omitted, is studied in [7, Appendix A], and then a Morita context is used to relate S with S_i.

Before doing likewise for T and T_i, it is convenient to recall certain facts regarding Morita contexts whose proofs may be discovered in [1]. As noted in [7], given a Morita context $\left[\begin{smallmatrix} \Lambda & V \\ W & \Gamma \end{smallmatrix}\right]$ there is one-to-one correspondence between

$$\mathcal{P}(\Lambda) = \{I \lhd \Lambda \mid I \text{ prime and } VW \not\subset I\}$$

and

$$\mathcal{P}(\Gamma) = \{I' \lhd \Gamma \mid I' \text{ prime and } WV \not\subset I'\}$$

given by $I \mapsto \theta(I) = \{\gamma \in \Gamma \mid V\gamma W \subset I\}$.

PROPOSITION 2.4 [**12**, LEMMA 4]. *Let $I \in \mathcal{P}(\Lambda)$. Then Λ/I is right nonsingular if and only if $\Gamma/\theta(I)$ is right nonsingular.*

Next, suppose that X_Λ is a *prime* module, which means that if $xI = 0$ for $0 \neq x \in X$ and $I \lhd \Lambda$ then $XI = 0$ (or, equivalently, that ann X_Λ = ass X_Λ). It follows that ann X is a prime ideal, J say.

LEMMA 2.5. *Suppose that X and J are as above and $0 \neq x \in X$. Let $U = \{v \in V \mid xvW = 0\}$, a submodule of V_Γ.*

(i) *If $J \notin \mathcal{P}(\Lambda)$ then $V/U = 0$;*

(ii) *if $J \in \mathcal{P}(\Lambda)$ then $(V/U)_\Gamma$ is a prime module with annihilator $\theta(J)$. Moreover, if X_Λ is uniform, or simple, then so too is $(V/U)_\Gamma$.* □

Consider now the Morita context relating T and T_i given by

$$\begin{bmatrix} T & TT^\circ f_i \\ f_i T^\circ T & T_i \end{bmatrix}.$$

We write θ_i rather than θ, and $\mathcal{P}_i(T)$ rather than $\mathcal{P}(T)$ when precision is required. The next result provides criteria, in a standard setting for J, for J to belong to $\mathcal{P}_i(T)$.

PROPOSITION 2.6. *Let T/K be a cyclic right T-module with $\operatorname{ann}(T/K)_T = J$. The following are equivalent.*

(i) $T^\circ f_i T^\circ \subset J$ (*i.e.*, $J \notin \mathcal{P}_i(T)$).

(ii) $T_{ii}{}^q \subset J$ *for some* q.

(iii) $T_{ii} \subset J$.

(iv) $f_i T^\circ \subset J$.

(v) $T^\circ f_i \subset J$.

(vi) $(I_0 \cap T) f_i (I_0 \cap T) \subset J$.

(vii) $T^\circ f_i \subset K$.

PROOF. Note that

$$(I_0 \cap T) f_i \subset T^\circ f_i = \sum_{j,k} T_{jk} f_i = \sum_j T_{ji} \subset T$$

and similarly, $f_i(I_0 \cap T) \subset f_i T^\circ \subset T$. Then the following implications are all clear: (vi) ⇐ (i) ⇐ {(iv) or (v)} ⇒ (iii) ⇒ (ii). Further implications will be considered one by one. Set $D = (I_0 \cap T) f_i (I_0 \cap T) \lhd T$.

(ii) ⇒ (vi). One gets $D^{q+1} \subset (I_0 \cap T)(f_i T^\circ f_i)^q (I_0 \cap T) \subset TT_{ii}{}^q T \subset J$ and so $D \subset J$.

(vi) ⇒ {(iv) and (v)}. Since $(I_0 \cap T) f_i \subset T$ and $(I_0 \cap T) \not\subset J$, then $(I_0 \cap T) f_i \subset J$ and so $(I_0 \cap T) f_i T^\circ \subset J$. Therefore $f_i T^\circ \subset J$. Similarly $T^\circ f_i \subset J$.

(v) ⇒ (vii) is clear since $J \subset K$.

(vii) ⇒ (vi). One has $D \subset T^\circ f_i T^\circ \subset K$, and so $D \subset \operatorname{ann} T/K = J$. □

COROLLARY 2.7. *If* $J \in \mathcal{P}_i(T)$, *then* $T_{ii} \not\subset \theta_i(J)$.

PROOF. $T_{ii}^3 \subset TT^\circ f_i T_{ii} f_i T^\circ T$ and $T_{ii}^3 \not\subset J$. Therefore $T_{ii} \not\subset \theta_i(J)$. □

LEMMA 2.8. *If* $J \in \mathcal{P}_i(T)$ *and* $0 \neq I' \lhd S_i$ *then* $I' \cap T_i \not\subset \theta_i(J)$.

PROOF. Suppose, to the contrary, that $I' \cap T_i \subset \theta_i(J)$. Now if $I = \{s \in S \mid f_i S^\circ S s S S^\circ f_i \subset I'\}$ then I is an ideal of S and $S_{ii} I' S_{ii} \subset I$, as is easily verified. Since $T^\circ \subset S^\circ$ it follows that

$$(f_i T^\circ T)(I \cap T)(TT^\circ f_i) \subset I' \cap T \subset \theta_i(J)$$

and so $I \cap T \subset J$. Since the setting is standard for J, $I = 0$ and so $S_{ii} I' S_{ii} = 0$. But S_i is prime and S_{ii} is a nonzero ideal of S_i [7, 2.22]. This provides a contradiction. □

The next few results concern T/J, viewed as a (T, R)-bimodule.

PROPOSITION 2.9. *Let* $X = T/K$ *be a cyclic prime right T-module, with* ann $X = J$. *Set* $X_i = (TT^\circ f_i + K)/K \subset T/K$. *In the Morita context of* 2.6

(i) *if* $J \notin \mathcal{P}_i(T)$ *then* $X_i = 0$; *and*

(ii) *if* $J \in \mathcal{P}_i(T)$ *then* X_i *has a natural right* T_i*-module structure and, as such, is prime with annihilator* $\theta_i(J)$. *Furthermore, if* X_T *is uniform, or simple, then so too is* $(X_i)_{T_i}$.

PROOF. Note that $X_i \simeq TT^\circ f_i/(TT^\circ f_i \cap K)$. Thus the result will follow directly from 2.5, applied to this Morita context, provided that, in the notation of 2.5, it is shown that $U = V \cap K$, where $V = TT^\circ f_i$ and $x = [1 + K]$. In this case $U = \{v \in TT^\circ f_i \mid v f_i T^\circ \subset K\}$.

(i) Here $T^\circ f_i \subset J$, by 2.6, and so $U = V = TT^\circ f_i \subset K$.

(ii) Again let $D = (I_0 \cap T) f_i (I_0 \cap T)$, and suppose $v \in U$.
Then $vD = v f_i D = v f_i (I_0 \cap T) f_i (I_0 \cap T) \subseteq v f_i T^\circ T \subset K$. But $D \not\subset J$, by 2.6, and X is prime; so $v \in K$ and thus $U \subset V \cap K$. On the other hand, $(K \cap TT^\circ f_i) f_i T^\circ \subset KT = K$ and so $V \cap K \subset U$. □

The next result involves the prime ideals $P_i = \operatorname{ann}_R f_i$. As in 2.9,

$$X_i = (TT^\circ f_i + K)/K.$$

PROPOSITION 2.10. *Each* X_i *is an R-submodule of* $(T/K)_R$ *with* $X_i P_i = 0$, *and there is a chain of right R-module inclusions*

$$0 \neq (I_0 \cap T) + K/K \subset \sum_{i=1}^{m}{}^{\oplus} X_i \subset T/K.$$

PROOF. Since the setting is standard for J, and $0 \neq I_0 \lhd S$, then $I_0 \cap T \not\subset J = \operatorname{ann} T/K$. Therefore $I_0 \cap T \not\subset K$ and so $0 \neq (I_0 \cap T) + K/K$. It is easily checked that X_i is a right R-submodule of T/K and, moreover, that the right R-module structure inherited from T/K coincides with that induced by the homomorphism $R \twoheadrightarrow f_i R \hookrightarrow T_i$ and the T_i-module structure of X_i. Therefore $X_i P_i = 0$. It remains to show that the sum ΣX_i is direct. Suppose $0 = \Sigma_1^m x_i$ with $x_i = t_i + K$ and $t_i \in TT^\circ f_i$. Setting $y = \Sigma t_i$, then $y \in K$; so $K \supset y T_{ii} = t_i T_{ii}$ and hence $x_i T_{ii} = 0$. If $J \in \mathcal{P}_i(T)$

then $x_i T_{ii} = 0$ in the prime T_i module X_i, yet, by 2.7, $T_{ii} \not\subset \theta_i(J) = \operatorname{ann} X_i$, and thus $x_i = 0$. On the other hand, if $J \notin \mathcal{P}_i(T)$ then $X_i = 0$ and again $x_i = 0$, as required. □

Before completing the description of X_i, we recall from [7, 2.2] that f_iSf_i is a prime right (and left) R-module with annihilator P_i (i.e., in the terminology used there, f_iSf_i is *right* (*and left*) *torsion-free* over f_iR).

PROPOSITION 2.11. *Suppose $J \in \mathcal{P}_i(T)$. Then*
(i) $\theta_i(J) \cap f_iR = 0$;
(ii) *X_i is a prime right R-module with annihilator P_i.*

PROOF. Note first that $T_i \subset f_iSf_i$ which is a torsion-free normalizing f_iR-bimodule. Therefore, if $0 \neq D \lhd f_iR$ then [**6**, 4.2] DT_i is an essential f_iR-subbimodule of T_i and there exists $0 \neq D' \lhd f_iR$ with $0 \neq T_iD' \subset DT_i$.

(i) If $\theta_i(J) \cap f_iR \neq 0$ then taking this to be D, it follows from the above that $DT_i \cap T_{ii}$ is essential in T_{ii} considered as an R-bimodule. By [7, 2.25], there exists $0 \neq I' \lhd S$ with $I' \cap T_{ii} \subset DT_i \cap T_{ii} \subset DT_i$, and then $0 \neq I'_{ii} \lhd S_i$ [7, 2.14] and $(I'_{ii} \cap T_i)T_{ii} \subset I' \cap T_{ii} \subset DT_i \subset \theta_i(J)$. This is impossible, by 2.7 and 2.8, proving $\theta_i(J) \cap f_iR = 0$.

(ii) Suppose that $0 \neq D \lhd f_iR$ and $x \in X_i$ with $xD = 0$. With D' as above, $xT_iD'T_i \subset xDT_i = 0$. Since X_i is a prime right T_i-module, $x = 0$. This shows that X_i, considered as an f_iR-module, is torsion-free, which is readily seen as equivalent to (ii). □

It is convenient to extract some information for the case where the setting is not standard.

THEOREM 2.12. *Let T be any intermediate normalizing extension of R, T/K a cyclic prime right T-module, and $J = \operatorname{ann}(T/K)$.*

(a) *There exists a finite set $Y_1, \ldots, Y_k$ of right R-submodules of T and an ideal J_0 of T such that*

(i) *$Y_i \supsetneq K$ for each i,*
(ii) $0 \neq J_0 + K/K \subset \Sigma_1^{\oplus k}(Y_i/K) \subset T/K$, *and*
(iii) *each Y_i/K is a prime right R-module.*

(b) *If $K = J$, then each Y_i is a (T, R)-subbimodule of T and $\Sigma^{\oplus} Y_i/J$ is an essential subbimodule of ${}_T(T/J)_R$.*

PROOF. (a) If the setting were standard for J, the nonzero $X_i \subset T/K$ of 2.9–2.11 have inverse images in T which can be chosen for the Y_i, and $J_0 = I_0 \cap T$ is as required. If the setting is not standard, it can be made so, using 2.2; and then appropriate modules may be chosen in the factor rings, with inverse images in T being as required.

(b) The first statement is clear from the construction of Y_i; and the second is trivial because $0 \neq J_0 + J/J \lhd T/J$ and so $J_0 + J/J$ is essential in ${}_T(T/J)_R$. □

This now gives the first of the connections between primes of R and T.

THEOREM 2.13 (CUTTING DOWN). *Let T be an intermediate normalizing extension of R and J a prime ideal of T. Then $J \cap R$ is a finite intersection of prime ideals with mutually isomorphic factor rings.*

PROOF. With the notation of 2.12(a), in the case when $K = J$, it is clear that $\text{rt.ann.}_T(J_0 + J/J) = \text{rt.ann.}_T(T/J) = J$ and so

$$J \cap R = \text{rt.ann.}_R\left(\sum_1^{k}{}^{\oplus}(Y_i/J)\right) = \bigcap_1^k \text{rt.ann.}_R(Y_i/J).$$

But $\text{rt.ann.}_R(Y_i/J)$ is a prime ideal, also by 2.12(a). The isomorphism of factor rings follows from the corresponding result [**5**, 2.12] for the prime ideal I of S (I being chosen as in 2.2), since each $\text{rt.ann.}_R(Y_i/J)$ is one of the prime ideals obtained from I (i.e., a prime ideal in $\text{conn}_R\, I$). □

REMARK. Since $\Sigma^{\oplus}\, Y_i/J$ is essential in ${}_T(T/J)_R$, by (2.12), it is easy to show that if ${}_TV_R$ is any subbimodule of T/J with V_R a prime module, then $\text{rt.ann.}_R(V)$ is one of the ideals $\text{rt.ann.}_R(Y_i/J)$. This observation shows that the set of primes $\text{rt.ann.}_R(Y_i/J)$ constructed in 2.12 is independent of which normalizing extension S contains T, and of the choices of I, the idempotents f_i, etc. Indeed, this set is also independent of the choice to consider $(T/J)_R$ rather than ${}_R(T/J)$: after reducing to a standard setting for J the set of primes obtained, on the left or right, is simply $\{P_i \mid T_{ii} \not\subset J\} = \{P_i \mid J \in \mathcal{P}_i(T)\}$. This proves

PROPOSITION 2.14. (i) *The prime ideals of R obtained from J in* 2.13 *are independent of the choices made in their construction.*

(ii) *If I is chosen as in* 2.2, *these primes are a subset of* $\text{conn}_R(I)$.

DEFINITION 2.15. The primes described above are said to be *connected* to J (and vice-versa); and the set of primes will be denoted by $\text{conn}_R(J)$. The minimal members of this set then comprise $\text{link}_R(J)$; they are the minimal primes of the semiprime ideal $J \cap R$ and are said to be *linked* to J. This description means that $\text{link}_R(J)$ is independent of choices too.

3. Incomparability and essentiality. This section studies the (R, R)-bimodule structure of T/J, and uses this to obtain the Essentiality Theorem and its consequence, the Incomparability Theorem.

Again it will be supposed, until specified otherwise, that $R \subset T \subset S$ provides a standard setting for the prime ideal J of T. Relabelling the idempotents f_i if necessary, we may assume that $\text{conn}_R\, J$ consists of $\{P_1, \ldots, P_k\}$ with $k \leqslant m$. Thus $J \in \mathcal{P}_i(T)$ if and only if $1 \leqslant i \leqslant k$. To simplify notation, we write J_i rather than $\theta_i(J)$; so J_i is a proper prime ideal of T_i if $1 \leqslant i \leqslant k$, and $J_i = T_i$ if $k < i \leqslant m$. The first result is similar to 2.10, and uses the decomposition $T^\circ = \Sigma^{\oplus m}_{i,j=1} T_{ij}$.

LEMMA 3.1. *The sum $\Sigma^k_{i,j=1}(T_{ij} + J/J)$ is a direct sum of (R, R)-subbimodules of T/J, and*

$$0 \neq (I_0 \cap T + J)/J \subset \sum_{i,j=1}^{k}{}^{\oplus}\left(T_{ij} + J/J\right) = (T^\circ + J)/J \subset T/J.$$

PROOF. Clearly $I_0 \cap T + J \subset T^\circ + J = \Sigma^m_{i,j=1} T_{ij} + J$. From 2.6 it follows that if $i > k$ then $T_{ij} + T_{ji} \subset J$ for all j; so $T^\circ + J = \Sigma^k_{i,j=1} T_{ij} + J$. The remainder of the proof is similar to that of 2.10, and so is omitted. □

Consider now the prime rings T/J and $(T^\circ + J)/J$. Since they both contain $(I_0 \cap T + J)/J$ as an ideal, these rings have the same Martindale right quotient ring.

PROPOSITION 3.2. *Each f_i is associated with an idempotent $\bar{f}_i$ of $Q(T/J)$ such that*
(i) $\bar{f}_i \neq 0$ *if and only if* $1 \leq i \leq k$;
(ii) $\{\bar{f}_i \mid 1 \leq i \leq k\}$ *is a family of orthogonal idempotents summing to* 1 *in* $Q(T/J)$;
(iii) *for* $1 \leq i \leq k$, $\bar{f}_i$ *centralizes* R *and* $\operatorname{ann}_R(\bar{f}_i) = P_i$.

PROOF. For each i, the restriction to $(I_0 \cap T + J)/J$ of the map $(T^\circ + J)/J \to (T^\circ + J)/J$ defined by $t + J \mapsto f_i t + J$ gives rise to $\bar{f}_i$ in $Q(T/J)$. Clearly $\bar{f}_i \bar{f}_j = 0$ (when $i \neq j$) or $\bar{f}_i$ (when $i = j$), and $\bar{f}_1 + \bar{f}_2 + \cdots + \bar{f}_m$ is the identity in $Q(T/J)$. Now $\bar{f}_i = 0$ if and only if $f_i(I_0 \cap T \cap A) \subset J$ for some $A \lhd T$ with $J \subsetneq A$. It now follows easily from 2.6 that $\bar{f}_i = 0 \Leftrightarrow f_i(I_0 \cap T)A \subset J \Leftrightarrow J \notin \mathcal{P}_i(T)$ (i.e., when $i > k$). Thus (i) and (ii) are established.

(iii) Since $Q(T/J)$ is a ring, it can be regarded as a bimodule over T/J, hence over $R/J \cap R$, and thus as an (R, R)-bimodule. Since each f_i centralizes R, so does $\bar{f}_i$. When $\bar{f}_i \neq 0$, $T_{ii} + J/J$ is right (and left) torsion-free over $f_i R$, by 2.11 (and its left-right dual), and this easily implies $\operatorname{ann}_R(\bar{f}_i) = P_i$. $\square$

For the rest of this section, unless otherwise specified, i is in $\{1,\ldots,k\}$, so $J \in \mathcal{P}_i(T)$. Next we relate $\bar{f}_i Q(T/J)\bar{f}_i$ to T_i/J_i. Recall that T_i, J and J_i are all subsets of $Q(S)$.

PROPOSITION 3.3. (i) $T_{ii} + J/J \cong T_{ii} + J_i/J_i$ *as rings and as* (R, R)-*bimodules*; (*ii*) *this isomorphism extends to an isomorphism* $\bar{f}_i Q(T/J)\bar{f}_i \cong Q(T_i/J_i)$.

PROOF. (i) Recalling that $J_i = \{x \in T_i \mid T^\circ f_i x f_i T^\circ \subset J\}$, it is evident that $T_{ii} \cap J \subset T_{ii} \cap J_i$. Furthermore, since $T_{ii} \cap J_i$ is a subset of T with $(I_0 \cap T)(T_{ii} \cap J_i)(I_0 \cap T) \subset T^\circ f_i J_i f_i T^\circ \subset J$, it follows that $T_{ii} \cap J = T_{ii} \cap J_i$, and now (i) follows easily.

(ii) It is sufficient to show how $\bar{f}_i Q(T/J)\bar{f}_i$ can be identified as the Martindale quotient ring of $T_{ii} + J/J$. For $q \in Q(T/J)$, there exists $0 \neq A/J \lhd T/J$ with $\bar{f}_i q \bar{f}_i(A/J) \subset f_i T^\circ + J/J$, so multiplication on the left by $\bar{f}_i q \bar{f}_i$, when restricted to $(f_i(I_0 \cap T)A(I_0 \cap T)f_i + J)/J$ induces an element $\tilde{q}$ of $Q(T_{ii} + J/J)$. If $\tilde{q} = 0$ then $\bar{f}_i q \bar{f}_i X = 0$ for some ideal $X/J \neq 0$ in $T_{ii} + J/J$, and then

$$\bar{f}_i q \bar{f}_i\,[(I_0 \cap T)A(I_0 \cap T)X(I_0 \cap T) + J/J] = 0,$$

so $\bar{f}_i q \bar{f}_i = 0$ in $Q(T/J)$. Thus $\bar{f}_i Q(T/J)\bar{f}_i$ embeds in $Q(T_{ii} + J/J)$, and this embedding is a ring homomorphism.

To see this is an isomorphism, let $q \in Q(T_{ii} + J/J)$, with $q(X/J) \subset T_{ii} + J/J$ where $J \subsetneq X \lhd T_{ii} + J$. Set $Y = (I_0 \cap T)(T_{ii} + J)X$ (so that $f_i Y \subset X$) and $Z = Y(T_{ii} + J)(I_0 \cap T)$. Then $Z \lhd T$, $Z \not\subset J$. We shall define a map $\hat{q}$: $Z/J \to T/J$.

Any z in Z may be written as a finite sum $\Sigma_j y_j t_j x_j$ with $y_j \in Y$, $t_j \in T_{ii} + J$ and $x_j \in I_0 \cap T$: then the image under $\hat{q}$ of $z + J$ is defined to be the coset $\hat{z} = \Sigma_j q(f_i y_j + J)t_j x_j$. To see that this is well defined, note that if $z \in J$ then

$$f_i z(I_0 \cap T) \subset T \quad \text{and} \quad (I_0 \cap T)f_i z(I_0 \cap T) \subset J,$$

whence $f_i z(I_0 \cap T) \subset J$. Then, for every $v \in (I_0 \cap T)(T_{ii} + J)$, each $t_j x_j v \in T_{ii} + J$ and so

$$\hat{z}v = \sum_j q(f_i y_j + J)t_j x_j v = q(f_i zv + J) = q(J) = 0.$$

Therefore, as an element of T/J, $\hat{z}$ annihilates $(I_0 \cap T)(T_{ii} + J)$ and thus $\hat{z} = 0$. Hence $\hat{q}$ is a well-defined function from $Z/J \to T/J$: trivially it is a right T-module homomorphism, and it is easily verified that $\overline{f}_i\hat{q}\overline{f}_i$ corresponds to q in the embedding of $\overline{f}_iQ(T/J)\overline{f}_i$ into $Q(T_{ii} + J/J)$. $\square$

REMARK 3.4. Under the isomorphism $\overline{f}_iQ(T/J)\overline{f}_i \cong Q(T_i/J_i)$, the subring $\overline{f}_iR + (T_{ii} + J/J)$ is isomorphic to T_i/J_i, since $T_{ii} + J/J \cong T_{ii} + J_i/J_i$, and we have simply adjoined the R-submodule generated by the identity element in each ring.

PROPOSITION 3.5. *If $J \subset W \lhd {}_R(T^\circ + J)_R$ with W/J essential in ${}_R(T^\circ + J/J)_R$, then*

(i) *W is essential in ${}_RT_R$ and*

(ii) *there exists $0 \neq B \lhd S$ with $B \cap T \subset W$.*

PROOF. It is standard that W must be essential in $T^\circ + J$; and T° essential in T, so (i) is proved. Now [7, 2.25] gives (ii). $\square$

PROPOSITION 3.6. *$Q(T/J)\overline{f}_i$ is right torsion-free, and $\overline{f}_iQ(T/J)$ is left torsion-free, as an R/P_i-module.*

PROOF. (Right) Suppose $q = q\overline{f}_i$ has $qK = 0$ with $P_i \subsetneq K \lhd R$. Since

$$(f_iT^\circ T + J)/J$$

is left torsion-free over f_iR (by the left-right dual of 2.10), $(Kf_iT^\circ + J)/J$ is essential in $(f_iT^\circ + J)/J$, and so

$$(Kf_iT^\circ + J/J) \oplus \left(\sum_{\substack{j=1 \\ j\neq i}}^{k} f_jT^\circ + J/J \right)$$

is essential in $T^\circ + J/J$. By 3.5,

$$B \cap T \subset Kf_iT^\circ + \left(\sum_{\substack{j=1 \\ j\neq i}}^{k} f_jT^\circ \right) + J \quad \text{for some } 0 \neq B \lhd S.$$

Then

$$\begin{aligned} q(B \cap T + J/J) &= q\overline{f}_i(B \cap T + J/J) \\ &\subset q\overline{f}_i(Kf_iT^\circ + J/J) + q\overline{f}_i\left(\sum_{\substack{j=1 \\ j\neq i}}^{k} f_jT^\circ + J/J \right) = 0, \end{aligned}$$

and so q is zero in $Q(T/J)$.

(Left) Suppose $q = \overline{f}_iq$ satisfies $Kq = 0$ with $P_i \subsetneq K \lhd R$. Since $q = \overline{f}_iq$, $qD \subset f_iT^\circ + J/J$ for an appropriately chosen ideal $D \neq 0$ of $T^\circ + J/J$. Then $KqD = 0$, with

$qD \subset f_iT^\circ + J/J$, implies $qD = 0$ since $f_iT^\circ + J/J$ is left torsion-free over f_iR. Hence $q = 0$. □

The next observation concerns the relationship between T_i and f_iR. The proof, which requires an analysis of subrings of $Q(S_i)$, is deferred to Appendix B (see B.5 of the Appendix).

LEMMA 3.7. *J_i is maximal among ideals J' of T_i with $J' \cap f_iR = 0$.*

COROLLARY 3.8. *Any nonzero ideal of $(T_{ii} + J/J) + \bar{f}_iR$ meets $\bar{f}_iR$ nontrivially.*

PROOF. The proof is immediate from 3.7 and 3.4. □

PROPOSITION 3.9. *With Q' denoting $Q(T/J)$, suppose $0 \neq q = \bar{f}_iq\bar{f}_j \in \bar{f}_iQ'\bar{f}_j$ and suppose $0 \neq A \lhd T/J$. Then $RqR \cap A \neq 0$.*

PROOF. Choose $0 \neq B \lhd T/J$ with $qB \subset T/J$. Then, with T'_{ii} denoting $(T_{ii} + J)/J$,

$$T'_{ii}q\left(BA \cap \bar{f}_j Q'\bar{f}_j\right) \subset A \cap Q'\bar{f}_j \cap \bar{f}_i Q' = A \cap \bar{f}_i Q'\bar{f}_j .$$

Now, by 3.8, $T'_{ii} \cap \bar{f}_iR = \bar{f}_iH \neq 0$ for some $H \lhd R$ and $T'_{jj}BAT'_{jj} \cap \bar{f}_jR = \bar{f}_jK \neq 0$ for some $K \lhd R$. Noting that $\bar{f}_jK \subset BA \cap \bar{f}_jQ'\bar{f}_j$, and using 3.6, it follows that

$$0 \neq HqK = H\bar{f}_i q\bar{f}_j K \subset T'_{ii}q\left(BA \cap \bar{f}_j Q'\bar{f}_j\right) \subset A \cap \bar{f}_i Q'\bar{f}_j ,$$

as desired. □

Next comes the main result of this section.

THEOREM 3.10 (ESSENTIALITY). *Let T be an arbitrary intermediate normalizing extension of a ring R. Let J be a prime ideal of T and $0 \neq A \lhd T/J$. Then A is essential in ${}_RQ(T/J)_R$ and in ${}_R(T/J)_R$.*

PROOF. Let $Q' = Q(T/J)$; so $Q' = \Sigma^{\oplus k}_{i,j=1}\bar{f}_iQ'\bar{f}_j$ by 3.2. By 3.9, $A \cap \bar{f}_iQ'\bar{f}_j$ is essential in $\bar{f}_iQ'\bar{f}_j$ for all i, j, from which it is standard that A is essential in ${}_RQ'_R$ and hence in ${}_R(T/J)_R$. □

COROLLARY 3.11. *Let S provide a standard setting for J. Then, for $W \lhd {}_RT_R$ with $W \supset J$, the following are equivalent*:

(i) *W/J is essential in ${}_R(T/J)_R$.*

(ii) *$W \supset J_1$ for some $J_1 \lhd T$, $J_1 \not\subset J$.*

(iii) *$W \supset B \cap T$ for some $0 \neq B \lhd S$.*

PROOF. (iii) ⇒ (ii) is clear; (ii) ⇒ (i) follows from 3.10, and (i) ⇒ (iii) follows from 3.5. □

COROLLARY 3.12 (INCOMPARABILITY). *Let T be an arbitrary intermediate normalizing extension of R, and let J, J' be ideals of T with J prime and $J \underset{\neq}{\subset} J'$. Then $J \cap R \underset{\neq}{\subset} J' \cap R$; indeed $J' \cap R$ is not contained in any prime ideal of R minimal over $J \cap R$.*

PROOF. J_1/J is essential in ${}_R(T/J)_R$ and so $(J_1 \cap (R + J))/J$ is essential in $R + J/J$. Let P be a minimal prime over $J \cap R$; so $P/J \cap R$ is not essential in $R/J \cap R$. Therefore $J + P/J$ is not essential in $R + J/J$. Hence $J + (J_1 \cap R) = (R + J) \cap J_1 \not\subset J + P$, and so $J_1 \cap R \not\subset P$. □

We conclude this section with an observation about $\operatorname{conn}_R(J)$. This set was seen in §2 to consist of all ideals of the form $\text{rt. ann.}_R(V)$ where ${}_TV_R$ is a subbimodule of T/J which is prime when considered as a right R-module. In fact, $\operatorname{conn}_R(J)$ can be identified solely in terms of $(T/J)_R$. To see this, define as M, the set of associated primes of an arbitrary module M_Λ, to be the set of all $\operatorname{ann}_\Lambda N$ for which N_Λ is a (nonzero) prime submodule of M.

PROPOSITION 3.13. $\operatorname{conn}_R(J) = \operatorname{ass}(T/J)_R = \operatorname{ass}_R(T/J)$.

PROOF. It may be assumed that S provides a standard setting for J. If $P_i \in \operatorname{conn}_R(J)$ then $T_{ii} + J/J$ is a prime right R-module with annihilator P_i, so $P_i \in \operatorname{ass}(T/J)_R$. Conversely, let $P \in \operatorname{ass}(T/J)_R$, and suppose $P = \operatorname{ann}_R(V)$ with V_R a prime submodule of T/J. As a submodule of $Q(T/J)_R$, $V \subset \Sigma_1^{\oplus k} V\bar{f}_i$. If $j \in \{1, \ldots, k\}$ is minimal with respect to $V \cap (\Sigma_1^j V\bar{f}_i) \neq 0$, then this intersection is a nonzero submodule of V_R which embeds into $Q(T/J)\bar{f}_jR$. Since V and $Q(T/J)\bar{f}_j$ are prime right R-modules, it follows that $P = P_j$ and thus $\operatorname{conn}_R(J) = \operatorname{ass}(T/J)_R$. The other assertion has a similar proof. □

4. Lying over and going up. In this section analogues of the Cohen-Seidenberg "lying over" and "going up" theorems are established for prime ideals of R and T. Then results related to maximum ideals and primitive ideals are obtained.

THEOREM 4.1 (LYING OVER). *Let T be an intermediate normalizing extension of R, and let P be a prime ideal of R. Then*

(i) *T contains an ideal J maximal with respect to $J \cap R \subset P$;*

(ii) *any such ideal J is prime, and P is a minimal prime over $J \cap R$.*

PROOF. (i) is a routine application of Zorn's lemma, and J is easily seen to be prime. To prove the rest of (ii) we may assume without loss of generality that J is in a standard setting. If P is not a minimal prime of $J \cap R$ then $P/J \cap R$ is essential in ${}_R(R/J \cap R)_R$, and if C is a complement of $R + J/J$ in ${}_R(T/J)_R$, then $(P + J/J) \oplus C$ is essential in ${}_R(T/J)_R$. By 3.11, S contains an ideal B with $B \cap T \not\subset J$ but $(B \cap T + J)/J \subset (P + J/J) \oplus C$. Then $(R + J) \cap (B \cap T + J) \subset P + J$ and so $R \cap (B \cap T + J) = R \cap (R + J) \cap (B \cap T + J) \subset R \cap (P + J) = P$, contradicting the maximality of J. □

COROLLARY 4.2 (GOING UP). *Let $P_1 \subset P_2$ be prime ideals of R and J_1 a prime ideal of T with $J_1 \cap R \subset P_1$. Then there exists a prime ideal J_2 of T with $J_1 \subset J_2$ and P_2 a minimal prime over $J_2 \cap R$.* □

Even in the case where S is a finite centralizing (or liberal) extension of R, this result is new, answering a question left open in [**15**].

It is useful to have some similar results for the primes of T and of S. If J is a prime ideal of T and I is an ideal of S maximal with respect to $I \cap T \subset J$ (as in 2.2) let us say that J, I form a *standard pair*.

THEOREM 4.3 (LYING OVER; T AND S). *Let $R \subset T \subset S$ with S a normalizing extension of R, and let J be a prime ideal of T. Then there is a standard pair J, I with I a prime ideal of S; and J is a minimal prime over $I \cap T$.*

PROOF. Only the last statement needs proof; and for it we may assume $I = 0$, using 2.2. Suppose that J' is a prime ideal of T with $J' \subset J$. Pick any idempotent f_i (as in §2) with $J \in \mathcal{P}_i(T)$. Then evidently $J' \in \mathcal{P}_i(T)$. Since $\theta_i(J') \subset \theta_i(J)$ and $\theta_i(J) \cap f_i R = 0$, then 3.7 applied to $\theta_i(J')$ gives $\theta_i(J') = \theta_i(J)$ and so $J' = J$. Thus J is minimal. □

COROLLARY 4.4 (GOING UP; T AND S). *Let $J_1 \subset J_2$ be prime ideals of T, and I_1 a prime ideal of S with $I_1 \cap T \subset J_1$. Then there exists a prime ideal I_2 of S with $I_1 \subset I_2$ and J_2 minimal over $I_2 \cap T$.* □

Next, we investigate the relationship between prime ideals P and J where P is a minimal prime over $J \cap R$, calling such a pair a *linked pair*.

THEOREM 4.5. *If T is an intermediate normalizingextension of R and P, J is a linked pair, then P is a maximal ideal of R if and only if J is a maximal ideal of T.*

PROOF. If J is maximal then 4.2 shows that P is maximal. Conversely suppose P is maximal. Using 2.2, we may suppose that the setting is standard for J. By 2.14(ii), P must be connected to the zero ideal of S and so, by [**5**, 5.11], S must be a simple ring. Now 3.11 applies to show that J is a maximal ideal of T. □

Before obtaining a corresponding result for primitive ideals, an extension of a result of Formanek and Jategaonkar [**4**] (for normalizing extensions) is required.

THEOREM 4.6. *Let T be an intermediate normalizing extension of R and X a simple right T-module. Then X_R is semisimple of finite length.*

Without loss of generality, we may suppose that $R \subset T \subset S$ is a standard setting for $J = \operatorname{ann}_T X$, and that $X = T/K$, with K a maximal right ideal. Application of 2.9–2.11 shows that $T/K = \Sigma_{i=1}^{\oplus m} X_i$ with each nonzero X_i a simple right T_i-module with annihilator $\theta_i(J) = J_i$. This, together with 2.8, shows that the chain $f_i R \subset T_i \subset S_i \subset f_i S f_i \subset Q(S_i)$ is appropriate for the theory in Appendix B. Then B.12 shows that X_i is semisimple of finite length as a right $f_i R$-module. Hence $(T/K)_R$ is semisimple of finite length. □

THEOREM 4.7. *If T is an intermediate normalizing extension of R and P, J is a linked pair, then P is right primitive if and only if J is right primitive.*

PROOF. Suppose J is right primitive, with $J = \operatorname{ann} X_T$ and X_T simple. By 4.4, $X_R = \Sigma^{\oplus} W_i$ with each $(W_i)_R$ simple. Therefore $J \cap R = \cap \operatorname{ann}_R W_i$. Therefore $P = \operatorname{ann}_R W_i$ for some i, and so P is right primitive.

Conversely, suppose P is right primitive. Choose I as in 2.2. By [**5**, 5.11], I is right primitive, say $I = \operatorname{ann}_S Y$ with Y a simple right S-module. By 4.6 (applied to R and S) Y_R is semisimple of finite length. Therefore Y_T has finite length. If $Q_1, \ldots, Q_k$ are the annihilators of the composition factors, then $\prod Q_i \subset \operatorname{ann}_T Y = I \cap T \subset J$. Thus, for some i, $Q_i \subset J$. Since the setting is standard for J, 4.3 shows that $J = Q_i$. □

Let $\mathbf{J}$ denote the Jacobson radical, and $\mathbf{P}$ the prime radical.

COROLLARY 4.8. *If T is an intermediate normalizing extension of R then $\mathbf{J}(R) = \mathbf{J}(T) \cap R$ and $\mathbf{P}(R) = \mathbf{P}(T) \cap R$.*

PROOF. These follow from "cutting down" (2.13) and "lying over" (4.1). The former also relies on 4.7. □

REMARK 4.9. Suppose that J, I form a standard pair of primes in T and S. It follows, from the results above, that J is maximal if and only if I is maximal, and J is right primitive if and only if I is right primitive. To see this, one simply chooses a prime ideal P linked with J. Note that P is connected to I and so $R/P \simeq R/P'$ for the minimal primes P' over $I \cap R$. Thus a double application of 4.5 or 4.7 (to P, J and P', I) gives the claimed results.

A similar argument applies to the results of §6 (to come), but will not be explicitly pointed out each time.

5. Nilpotency. Easy examples, involving triangular matrix subrings of the examples described in [**5** and **7**] show that when $R \subset T \subset S$ as usual, and I is a prime ideal of S, then $I \cap T$ need not be semiprime (as is $I \cap R$). This section provides some information about the nature of the ring $T/I \cap T$. As the examples would suggest, it has a prime radical which is nilpotent, with index at most the number n of normalizing generators of S, and it has at most n minimal primes. As indicated at the end of this section, there are some unanswered questions remaining.

First comes a result about prime normalizing extensions.

PROPOSITION 5.1. *Let S be a prime normalizing extension of R, with n normalizing generators. Let X be both a nilpotent subring and an R-subbimodule of S. Then $X^n = 0$.*

PROOF. Suppose that $X^t = 0$. In $Q(S)$, the idempotents f_i sum to 1; so $X^j \subset \Sigma_{i=1}^{\oplus m} X^j f_i$ for each j. Thus it suffices to show $X^n f_i = 0$ for each i.

Fixing i, and setting $X^0 = S$, consider the chain $Sf_i = X^0 f_i \supset X f_i \supset X^2 f_i \supset \cdots \supset X^t f_i = 0$ of subbimodules of the normalizing $(R, f_i R)$-bimodule Sf_i. Let k be the least integer for which $X^k f_i / X^{k+1} f_i$ is not faithful over $f_i R$. Then if ρ is the rank $\rho_{f_i R}$ defined in [**6**, 3.1], $k \leqslant \rho(Sf_i) \leqslant n$ by [**6**, 3.3 and 3.5]. Also, if $j \geqslant k$, then $X^j f_i D_j \subset X^{j+1} f_i$ for some $D_j \lhd R$ with $P_i \subsetneq D_j$, since $X^j f_i / X^{j+1} f_i$ is unfaithful. Therefore

$$X^k f_i D_k D_{k+1} \cdots D_{t-1} \subset X^t f_i = 0.$$

However, $0 \neq f_i D_k D_{k+1} \cdots D_{t-1}$ is an ideal of $f_i R$, and Sf_i is right torsion-free over $f_i R$. Therefore $X^k f_i = 0$; and so $X^n = 0$ as claimed. □

THEOREM 5.2. *Suppose that T is an intermediate extension with $R \subset T \subset S$ and that I is a prime ideal of S. Then the prime radical of $T/I \cap T$ is nilpotent of index at most n.*

PROOF. Without loss of generality, suppose $I = 0$. Then any nilpotent ideal X of T satisfies $X^n = 0$, by 5.1. A standard argument shows that the sum of all nilpotent ideals of T is nilpotent, and coincides with the prime radical of T. □

COROLLARY 5.3. *If $R \subset T \subset S$ is an arbitrary intermediate normalizing extension then $\mathbf{P}(T)^n \subset \mathbf{P}(S) \cap T \subseteq \mathbf{P}(T)$.*

PROOF. This follows directly from "lying over" (4.3) and 5.2. □

There are corresponding results concerning the Jacobson radical.

PROPOSITION 5.4. *Suppose that $R \subset T \subset S$ with S a right primitive normalizing extension of R. Then T has at most n minimal primes, each of which is right primitive; and $\mathbf{J}(T)^n = 0$.*

PROOF. The proof of 4.7 shows that the minimal primes of T are amongst the primitive ideals $Q_1, \ldots, Q_k$, with $k \leqslant n$. Then $\mathbf{J}(T) = \mathbf{P}(T)$ and so $\mathbf{J}(T)^n = 0$ by 5.2. $\square$

COROLLARY 5.5. *Let $R \subset T \subset S$ be an arbitrary intermediate normalizing extension. Then $\mathbf{J}(T)^n \subset \mathbf{J}(S) \cap T \subseteq \mathbf{J}(T)$.*

Comparison of 5.2 and 5.4 suggests that, in the notation of 5.2, $I \cap T$ should have at most n minimal primes. To show this, we use the "primitivity machine" of [**13**].

It is shown in [**13**] that given a ring, K say, there is an overring $K^\dagger$ with the property that, if P is any prime ideal of K, then $P^\dagger = PK^\dagger$ is a primitive ideal of $K^\dagger$. Other details required will be quoted as needed; but note that, simultaneously, one can form $R^\dagger$, $T^\dagger$ and $S^\dagger$; and the chain $R^\dagger \subset T^\dagger \subset S^\dagger$ gives again an intermediate normalizing extension, with $S^\dagger$ having n normalizing generators over $R^\dagger$.

THEOREM 5.6. *Suppose that T is an intermediate normalizing extension with $R \subset T \subset S$ and that I is a prime ideal of S. Then there are at most n primes of T minimal over $I \cap T$.*

PROOF. Without loss of generality suppose $I = 0$. Using the "primitivity machine", note that $S^\dagger$ is primitive and so, by 5.4, $T^\dagger$ has no more than n minimal primes.

Let $\{Q_\gamma \mid \gamma \in \Gamma\}$ be the primes of T; so $\bigcap Q_\gamma = \mathbf{P}(T)$. By [**13**, 1.4], $\bigcap Q_\gamma^\dagger = \mathbf{P}(T)^\dagger$ and, of course, each $Q_\gamma^\dagger$ is primitive. Thus $\mathbf{P}(T)^\dagger = \bigcap Q_\gamma^\dagger \supset \mathbf{P}(T^\dagger)$. On the other hand, $(\mathbf{P}(T))^n = 0$ and so, by [**13**, 1.4], $(\mathbf{P}(T)^\dagger)^n = 0$. Therefore $\mathbf{P}(T)^\dagger \subset \mathbf{P}(T^\dagger)$ and so $\mathbf{P}(T)^\dagger = \mathbf{P}(T^\dagger)$.

Now $T^\dagger/\mathbf{P}(T^\dagger)$ has at most n minimal primes, say $H_1, \ldots, H_h$; and each is the annihilator of its complement. Moreover, $T^\dagger/\mathbf{P}(T^\dagger) = T^\dagger/\mathbf{P}(T)^\dagger \simeq (T/\mathbf{P}(T))^\dagger$. Therefore, by [**13**, 1.5], $H_i \cap (T/\mathbf{P}(T)) = H_i'$ say, is a prime ideal. And since $\bigcap H_i = 0$, then $\bigcap H_i' = 0$. Thus $T/\mathbf{P}(T)$ has at most n minimal primes. $\square$

This is a convenient point at which to comment on the parallels between the behaviours of the pairs R, T and T, S. The results of this section show that, even though $I \cap T$ need not be semiprime (as is the case for $J \cap R$), nevertheless there are restrictions upon its prime radical. And, in both cases, there are at most n minimal primes involved. Likewise the results of §4 give parallel results.

What remains unclear at present is the validity of an "incomparability" theorem for the pair T, S. This could be the statement:

(a) If I and I' are ideals of S with $I \subsetneqq I'$ with I prime, and if J is a minimal prime over $I \cap T$, then $I' \cap T \not\subset J$.

It is not difficult, using the earlier results, to see that this is equivalent to each of the following statements:

(b) If S is prime and J is a minimal prime of T then the setting is standard for J (cf. 4.3).

(c) If S is prime and J is a minimal prime of T then J is not essential in ${}_RT_R$ (cf. 2.1).

The use of $S^\dagger$ again shows that it would suffice to prove any of these in the case when S is primitive. Also it is perhaps worth noting that when S is prime, the setting is standard for at least one of the minimal primes of T. (Otherwise one could deduce that $\mathbf{P}(T)$ is both nilpotent and essential in ${}_RT_R$. This would contradict R being semiprime.)

6. Chain conditions. Suppose that $R \subset T \subset S$ is an intermediate normalizing extension and that P, J is a linked pair of prime ideals of R and T. In this section we show that certain chain conditions pass between the two rings R/P and T/J. Indeed, as noted in 4.9, they also pass between these two rings and the ring S/I, when I is chosen as in 2.2. *This notation is fixed henceforth.*

Note first that both $R/I \cap R$ and $R/J \cap R$ are finite subdirect sums of rings isomorphic to R/P. So if one of $R/I \cap R$, $R/J \cap R$, and R/P is right nonsingular, or right Goldie, or right Noetherian, then so are the other two.

THEOREM 6.1. *Let T be an arbitrary intermediate normalizing extension of R, and P, J a linked pair of primes. Then R/P is right nonsingular if and only if T/J is right nonsingular.*

PROOF. Without loss of generality, we may suppose $I = 0$. Using the notation of §2 and 3, it follows that $P = P_i$ for one of the idempotents f_i such that $J \in \mathcal{P}_i(T)$. Let Z_i denote the inverse image in T_i of the singular ideal of T_i/J_i, where $J_i = \theta_i(J)$. By 3.7 and B.15 (Appendix B), f_iR is nonsingular if and only if $Z_i = J_i$. Thus R/P is nonsingular if and only if T_i/J_i is nonsingular, and so the result follows from 2.4 applied to the Morita context above 2.6. □

Next it is shown that having a finite uniform dimension passes between T/J and R/P. The second part of the following proof is similar to that of the comparable result for intermediate subrings of centralizing extensions, due to Stewart [**16**].

THEOREM 6.2. *Let T be an arbitrary intermediate normalizing extension of R, and P, J a linked pair of primes. Then* $\text{u. dim.}(T/J)_T < \infty$ *if and only if* $\text{u. dim.}(R/P)_R < \infty$.

PROOF. We may assume $R \subset T \subset S$ provides a standard setting for J, and that the idempotents f_i are labelled so that $\operatorname{conn}_R(J) = \{P_1, \ldots, P_k\}$. Let I_0 denote the ideal of S described in §2.

(i) Assume that $\text{u. dim.}(T/J)_T < \infty$. For each i with $1 \leqslant i \leqslant k$, there is, by 3.2 and 3.4, a Morita context

$$\begin{bmatrix} T/J & (TT^\circ + J/J)\overline{f}_i \\ \overline{f}_i(T^\circ T + J/J) & T_i/J_i \end{bmatrix}.$$

By [**1**, Theorem 2], $\text{u. dim.}(T_i/J_i)_{T_i} = \text{u. dim.}\ \overline{f}_i(T^\circ T + J/J)_T \leqslant \text{u. dim.}(T/J)_T$. Now B.13 of Appendix B gives $\text{u. dim.}(f_iR_{f_iR}) < \infty$. Since $R/P \cong f_iR$, $(R/P)_R$ has finite uniform dimension.

(ii) Now assume u. dim.$(R/P)_R$ is finite. It will be shown that u. dim.$(T/J)_T =$ u. dim.$(I_0 \cap T + J/J)_T \leqslant$ u. dim.$(I_0 \cap T + J/J)_R \leqslant n^3($u. dim.$(R/P)_R)$, and clearly only the last inequality needs proof. From 3.1,

$$I_0 \cap T + J/J \subset \sum_{i,j=1}^{k}{}^{\oplus} T_{ij} + J/J \subset T/J.$$

Setting $X_{ij} = T_{ij} + J/J$, the sum $\Sigma_{i,j=1}^{\oplus k} X_{ij}$ is a direct sum of (R, R)-bimodules. It is clearly sufficient to prove that u. dim.$(X_{ij})_R \leqslant n[$u. dim.$(R/P)_R]$ whenever $i, j \leqslant k$.

Note that $f_iR \cong R/P \cong f_jR$, as rings, so R/P and f_jR have the same right uniform dimension. Inside $Q(S)$, the inclusion $T_{ij} \subset f_iSf_j = \Sigma_{v=1}^{n} f_i a_v f_j R$ leads to an inclusion $T_{ij} + J/J \hookrightarrow f_iSf_j/T_{ij} \cap J$ of (f_iR, f_jR)-bimodules. Denote $f_i a_v f_j$ by x_v $(1 \leqslant v \leqslant n)$ and set $T_{ij} \cap J = Y$, this being an (f_iR, f_jR)-subbimodule of f_iSf_j. With $^{-}$ denoting the coset in f_iSf_j/Y, choose $\Gamma = \Gamma(i, j)$, a subset of $\{1, \ldots, n\}$ maximal with respect to the property that $\bar{F} = \Sigma_{\gamma \in \Gamma} Rf_i \bar{x}_\gamma$ is a free left Rf_i-module with basis $\{\bar{x}_\gamma \mid \gamma \in \Gamma\}$. If $\{r_\gamma \mid \gamma \in \Gamma\} \subset R$ and $\Sigma_{\gamma \in \Gamma} r_\gamma x_\gamma \in T_{ij} \cap J$, then from the definition of Γ it follows that $r_\gamma f_i = 0$ for each γ. In particular, $F = \Sigma_{\gamma \in \Gamma} Rx_\gamma = \Sigma_{\gamma \in \Gamma} Rf_i x_\gamma$ is a free left Rf_i-module of rank $|\Gamma|$, and $F \cap J = 0$.

For $1 \leqslant v \leqslant n$, set $D_v = \{r \in R \mid rf_i x_v \in F + Y\}$, this being an ideal of R since $f_i x_v$ normalizes R. It follows (from the definition of Γ) that $P_i \subsetneq D_v$, so $P_i \subsetneq D = \bigcap_1^n D_v \lhd R$ and $Df_iSf_j \subset F + Y$. Since f_iSf_j is a torsion-free normalizing (f_iR, f_jR)-bimodule, there exists [**5**, 1.5] $D' \lhd R$ with $P_j \subsetneq D'$ and $f_iSf_jD' \subset Df_iSf_j$. Then $(T_{ij} + J/J)D' \subset F + Y + J/J \cong F/F \cap J = F$, this being an embedding of (f_iR, f_jR)-bimodules and (R, R)-bimodules. But X_{ij} is right torsion-free over f_jR (2.11), so $X_{ij}D'$ is essential in X_{ij} as a right f_jR-module. Therefore u. dim.$(X_{ij})_R =$ u. dim.$(X_{ij}D')_R \leqslant$ u. dim.$(F_R) = |\Gamma|$u. dim.$[R/P]_R$, the last equality following because F_R is a direct sum of $|\Gamma|$ copies of f_jR. □

The next result encompasses both a result of Lanski [**10**] and one of Stewart [**16**].

THEOREM 6.3. *If P, J is a linked pair of prime ideals then R/P is right Goldie if and only if T/J is right Goldie.*

PROOF. Since prime right Goldie rings are characterized as prime right nonsingular rings with finite right uniform dimension, this follows at once from 6.1 and 6.2. □

In the case where R/P and T/J are right Goldie, Theorem 6.2 can be strengthened by showing that u. dim.$(T/J)_T$ is an integer multiple of u. dim.$(R/P)_R$, thus providing another instance of the validity of the "additivity principle". First, some additional notation is required.

NOTATION 6.4. With $R \subset T \subset S$ providing a standard setting for J, and P, J a linked pair, suppose $\operatorname{conn}_R(J) = \{P_1, \ldots, P_k\}$. Denote by $R^{\#}$ the subring $\Sigma_{i=1}^{\oplus k} f_iR$ of $Q(S)$. Via the homomorphism induced by $f_i \mapsto \bar{f}_i$, $R^{\#}$ embeds as a subring of $Q(T/J)$ with the same unity, and so $T^{\circ} + J/J$ can be regarded as an $R^{\#}$-bimodule. Since $T^{\circ} + J/J = \Sigma_{i,j=1}^{\oplus k}(T_{ij} + J)/J$ and each $T_{ij} + J/J$ is torsion-free as an (f_iR, f_jR)-bimodule, from 3.6, $T^{\circ} + J/J$ is torsion-free as an $R^{\#}$-bimodule.

Next, a preparatory lemma is needed.

LEMMA 6.5. *In the embedding of $R^{\#}$ into $Q(T/J)$, a regular element c of $R^{\#}$ is not a zero-divisor in $Q(T/J)$.*

PROOF. Since $Q(T/J)$ is the Martindale quotient ring of $T^\circ + J/J$, it is sufficient to show that $\text{rt. ann.}_{(T^\circ + J/J)}(c) = \text{lt. ann.}_{(T^\circ + J/J)}(c) = 0$. Furthermore, we may write $c = \sum_1^k c_i$ with c_i a regular element of $f_i R$.

Suppose first that $t \in T^\circ$ and $c(t + J) = 0$. By 3.1,

$$(T^\circ + J)/J = \sum_{i,j=1}^{k}{}^{\oplus} (T_{ij} + J)/J,$$

so we may write $t + J = \sum_{i,j}(t_{ij} + J)$ with $t_{ij} \in T_{ij}$. Then $c(t + J) = 0$ implies $c_i t_{ij} \in J$ for each i and j.

Suppose i and j are chosen, and let $\Gamma = \Gamma(i, j)$, $\bar{F}$, and $0 \neq D' \lhd f_j R$ be chosen as in the proof of 6.2(ii). Then $\bar{F}$ is an $(f_i R, f_j R)$-bimodule, free as an $f_i R$-module, with an $(f_i R, f_j R)$-normalizing basis $\{\bar{x}_\gamma \mid \gamma \in \Gamma\}$. Furthermore, as in 6.2, $(T_{ij} + J/J)D' \subset \bar{F}$. For any d in D', write $(t_{ij} + J)d = \sum_\gamma r_\gamma \bar{x}_\gamma$ with $r_\gamma \in f_i R$. Then $c_i(t_{ij} + J)d = 0 = \sum_\gamma c_i r_\gamma \bar{x}_\gamma$, and since ${}_{f_i R}\bar{F}$ is free, $c_i r_\gamma = 0$ for each γ. Since c_i is regular in $f_i R$, each $r_\gamma = 0$: therefore $(t_{ij} + J)D' = 0$, and since $(T_{ij} + J)/J$ is right torsion-free over $f_j R$, $t_{ij} + J = 0$.

Thus $t + J = 0$ whenever $c(t + J) = 0$, and so c is right regular in $Q(T/J)$. A symmetrical argument (including a left-hand version of 6.2) completes the proof. □

Now comes the "additivity principle".

THEOREM 6.6. *Assume that P, J is a linked pair of prime with T/J and (equivalently) R/P right Goldie. Then*

(i) *the embedding of $R^{\#}$ into $Q(T/J)$ gives an embedding of $Q_c(R^{\#})$ into $Q_c(T/J)$, where Q_c is the classical right quotient ring, and*

(ii) $\text{u. dim.}(T/J)_T = z \cdot \text{u. dim.}(R/P)_R$ *where z is an integer divisible by $k = |\text{conn}_R(J)|$.*

PROOF. (i) As usual, a standard setting for J may be assumed, and I_0 is the ideal of S described previously. Then each of $I_0 \cap T + J/J$, $R^{\#} + T^\circ/J$, and T/J are orders in $Q_c(T/J)$. By 6.5, any regular element of $R^{\#}$ remains regular as an element of $R^{\#} + T^\circ/J$, and hence is a unit in $Q_c(T/J)$, so $Q_c(R^{\#})$ embeds in $Q_c(T/J)$.

(ii) It now follows by standard arguments (see [**18**, Lemma 1 or **9**, 3.8]) that $\text{u. dim.}[T/J]_T$ is a multiple of the composition length of $Q_c(R^{\#})$, and the latter equals k $(\text{rank}(R/P)_R)$. □

REMARKS. (1) This resolves a question left open in [**9**].

(2) If P, J and I are primes of R, T and S, respectively, with J, I forming a standard pair and P linked to both J and I, then (when R/P, T/J and S/I are right Goldie) $\text{rank}(R/P)_R$ divides both $\text{rank}(T/J)_T$ and $\text{rank}(S/I)_S$. It need not be true that $\text{rank}(T/J)$ divides $\text{rank}(S/I)$. For example, let A be any commutative domain

with a surjective homomorphism $\theta: A \twoheadrightarrow A$ with nonzero kernel K. Take $S = M_3(A)$, R the subring $\{\operatorname{diag}(a, a, \theta(a)) \mid a \in A\}$ and T the subring consisting of all elements of the form

$$\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & \theta(d) \end{bmatrix} \quad \text{with } a, b, d \in A \text{ and } c \in K.$$

Then taking $J = 0$ (in T) and $I = 0$ (in S), it is easily verified that $\operatorname{rank}(T_T) = 2$ and $\operatorname{rank}(S_S) = 3$.

Before considering the passage of other chain conditions, and Krull dimension, between T/J and R/P, a preparatory result is required.

PROPOSITION 6.7. *Suppose that $T/\operatorname{ann} M$ is right or left Noetherian, and that M_T is a Noetherian module with $\kappa(M) = \alpha$. Then M_R is Noetherian with $\kappa(M_R) = \alpha$.*

PROOF. Suppose by Noetherian induction that M is a counterexample of minimal possible Krull dimension and that no proper factor module of M_T is also a counterexample. Any critical submodule of M_T will also have this property, so suppose M is indeed critical. Since $T/\operatorname{ann} M$ is Noetherian, M has a prime cyclic submodule M', with prime annihilator, J say. Again, we may suppose $M = M'$; and without loss, we may also suppose the setting to be standard for J.

Suppose, then, that $M = T/K$. By 2.9, $0 \neq (I_0 \cap T) + K/K \subset \Sigma^{\oplus} X_i \subset T/K = M$ with each X_i either zero or a prime uniform T_i-module. Suppose $0 \neq Y \lhd (X_i)_{T_i}$; we will show that $(X_i/Y)_R$ is Noetherian, with $\kappa(X_i/Y)_R < \alpha$.

To see this, say $Y \simeq Z/K$, with $K \subset Z \underset{\neq}{\subset} TT^\circ f_i + K$. There is then a well-defined multiplication $Y \cdot T^\circ \subset T/K$ under which

$$Y(I_0 \cap T) \subset Y \oplus \sum_{\substack{j=1 \\ j \neq i}}^{m} X_j$$

(since $YT_{ii} \subset Y$). Now

$$X_i/Y \simeq \Big(X_i \oplus \sum_{j \neq i} X_j\Big) \Big/ \Big(Y \oplus \sum_{j \neq i} X_j\Big) \subseteq M \Big/ \Big(Y \oplus \sum_{j \neq i} X_j\Big),$$

which is a homomorphic image of $(M/Y(I_0 \cap T))_R$. By induction, the latter is Noetherian, with Krull dimension $< \alpha$, so the same is true of $(X_i/Y)_R$.

Considering X_i as an R-module, or equivalently as an f_iR-module, B.13 of Appendix B shows that $\operatorname{u.dim.}(X_i)_R < \infty$ and that any essential submodule E_R contains a nonzero T_i-submodule, so from the preceding paragraph it follows that $(X_i/E)_R$ is Noetherian and has Krull dimension $< \alpha$. Therefore, by [**4**, Lemma 1], $(X_i)_R$ is Noetherian and $\kappa(X_i)_R \leqslant \alpha$. Now both $M/M(I_0 \cap T)_R$ and $M(I_0 \cap T)_R$ are Noetherian, with Krull dimension $\leqslant \alpha$, so the same is true of M_R. This completes the proof. □

THEOREM 6.8. *If P, J is a linked pair of primes then R/P is right Noetherian with Krull dimension α if and only if the same is true of T/J.*

PROOF. Without loss, suppose the setting is standard for J. Now if R/P is right Noetherian then so are R (as noted earlier), S_R $(= \sum_1^n a_i R)$, and $(T/J)_R$. Thus T/J is right Noetherian. Whereas if T/J is right Noetherian then so is $(T/J)_R$ (by 6.7), and hence $R/J \cap R$ and R/P are right Noetherian. The Krull dimension is dealt with similarly. $\square$

Another immediate consequence of 6.7 is the following, which is also, however, an easy consequence of the corresponding result for R, S.

COROLLARY 6.9. *R is right Noetherian or right Artinian if and only if the same is true of T.*

Appendix B. This section concerns a chain $R \subset T \subset S \subset M \subset Q = Q(S)$ where R and S are both prime rings, T an intermediate ring, M a normalizing R-bimodule with generators $x_1, \ldots, x_n$, and Q, the Martindale ring of right quotients of S, is a (right and left) torsion-free R-bimodule. Moreover, J is a prime ideal of T with $J \cap R = 0$.

The results obtained here are all used in earlier sections for the chain described in §2, viz. $f_iR \subset T_i \subset S_i \subset f_iSf_i \subset Q(S_i) = f_iQ(S)f_i$; and the statements of such "applications" will be given in that earlier notation. That apart, the notation will remain as in the first paragraph above.

This appendix is a continuation of Appendix A of **[7]**, the approach taken here being based on ideas from **[9, 15,** and **16]**. Now in Appendix A it was shown that the embedding $R \subset S$ extends to an embedding of NR (the normal closure of R) into Q. With $\mathfrak{M}$ denoting $\{q \in Q \mid qR = Rq\}$, and $\mathcal{C}$ the subring of Q generated by $\mathfrak{M}$ and R, each $H \lhd {}_RQ_R$ gives rise to a certain subset $\overline{H}$ of $\mathcal{C}$ (defined in **[7]**). The relevant properties are as follows.

PROPOSITION B.1. *With H, K and L denoting subbimodules of ${}_RS_R$, and $\mathfrak{M}_H$ denoting $\overline{H} \cap \mathfrak{M}$,*

(i) *if $q \in \overline{H}$ then $qY + Y'q \subset H$ for some nonzero ideals Y and Y' of R, and if $q \in \mathcal{C}$ satisfies $YqY' \subset H$ for $0 \neq Y$, $Y' \lhd R$, then $q \in \overline{H}$;*

(ii) *$\overline{H} = \mathfrak{M}_H R$, this being an R-bimodule and an NR-bimodule;*

(iii) *$\overline{H}$ is a free right and left NR-module of finite rank, and a finite NR-normalizing basis may be found in $\mathfrak{M}_H$. Any maximal R-independent subset of $\mathfrak{M}_H$ comprises such a basis;*

(iv) *if $HK \subset L$, then $\overline{H}\overline{K} \subset \overline{L}$. In particular, $\overline{T} = \mathfrak{M}_T R$ is a subring of $\overline{S} = \mathfrak{M}_S R$ containing $\mathfrak{M}_R R = NR$, and $\overline{B} \lhd \overline{T}$ whenever $B \lhd T$.*

PROPOSITION B.2. *$\mathfrak{M}_J R$ is maximal among ideals I of $\mathfrak{M}_T R$ satisfying $I \cap T = J$.*

PROOF. $J \subset M \subset \mathcal{C}$ so $J \subset \overline{J} \cap T = \mathfrak{M}_J R \cap T$, and if $x \in \mathfrak{M}_J R \cap T$ then $0 \neq Yx \subset J$ (by torsion-freeness of Q, and B.1), so J is essential in ${}_R(T \cap \mathfrak{M}_J R)_R$. By **[6**, 2.2], there exists $0 \neq D \lhd R$ with $D(T \cap \mathfrak{M}_J R) \subset J$. Since $D \not\subset J$ (because $J \cap R = 0$), it follows that $T \cap \mathfrak{M}_T R \subset J$, so $J = T \cap \mathfrak{M}_J R$.

Suppose now that $\mathfrak{M}_J R \subset L \lhd \mathfrak{M}_T R$ with $L \cap T \subset J$. Then for x in L $(\subset \mathfrak{M}_T R)$ there exists $0 \neq Y \lhd R$ with $Yx \subset T \cap L \subset J$; hence $x \in \overline{J} = \mathfrak{M}_T R$ and thus $\mathfrak{M}_J R = L$. $\square$

COROLLARY B.3. *$\mathfrak{M}_J R$ is a prime ideal of $\mathfrak{M}_T R$ and $\mathfrak{M}_J R \cap NR = 0$.*

PROOF. That $\mathfrak{M}_J R$ is prime follows from B.2. If $x \in \mathfrak{M}_J R \cap NR$, then there exists $0 \neq Y \lhd R$ with $Yx \subset J \cap R = 0$ and since ${}_R Q_R$ is torsion-free, $x = 0$. □

PROPOSITION B.4. *If $J \underset{\neq}{\subset} B \lhd T$, then $B \cap R \neq 0$.*

PROOF. Since $J \underset{\neq}{\subset} B \subset T \cap \mathfrak{M}_B R$, $\mathfrak{M}_J R$ is a proper subset of $\mathfrak{M}_B R \lhd \mathfrak{M}_T R$. Since $\mathfrak{M}_T R$ is a normalizing extension of NR, the "incomparability" theorem [**5**, 5.10] implies $\mathfrak{M}_B R \cap NR \neq 0$. If $0 \neq x \in \mathfrak{M}_B R \cap NR$, then $0 \neq Dx \subset B \cap R$ for some $D \lhd R$. □

APPLICATION B.5. With notation as in 3.7, J_i is maximal among ideals of T_i meeting $f_i R$ at zero.

LEMMA B.6. *Let $b_1, \dots, b_v$ be any NR-basis for $\mathfrak{M}_T R$ whose elements normalize R. Then there exist nonzero ideals D and F of R for which $\Lambda = \sum_1^v F b_i$ is a subring (without 1) of T containing TDT.*

PROOF. Since $\mathfrak{M}_T R$ is a ring, each product $b_i b_j$ may be written $b_i b_j = \sum_{k=1}^v \rho_{ijk} b_k$ with $\rho_{ijk} \in NR$. Then $F_1 \rho_{ijk} \subset R$ (for all i, j, k) for some $0 \neq F_1 \lhd R$. Choose $F_{1j} \lhd R$ with $b_j F_{1j} = F_1 b_j$ ($\neq 0$), for each j, and $Z \lhd R$ with $0 \neq Z b_j \subset T$ (using B.1(i)); and set $F = Z \cap (\bigcap_{j=1}^v F_{1j})$. Then it follows that $F b_i F b_j \subset \sum_{k=1}^v F b_k \subset T$, so $\Lambda = \sum_1^v F b_i$ is a ring. Since ${}_R NR_R$ is uniform, the (R, R)-bimodule ranks of T, $\mathfrak{M}_T R$ and Λ all equal v, so Λ is essential in ${}_R T_R$. Using the torsion-freeness of ${}_R Q_R$ and applying [**6**, 2.2] (twice) there exist D, $D' \lhd R$ with $0 \neq DT \subset TD' \subset \Lambda$, and so $TDT \subset \Lambda$. □

REMARK. Suppose that $H_1 \subset H_2 \subset \cdots \subset H_n$ was a finite chain of subrings (with or without 1) of S with $RH_iR \subset H_i$. From B.1, a chain $B_1 \subset B_2 \subset \cdots \subset B_n$ of finite subsets of $\mathfrak{M}_S$ could be chosen with B_i an R-normalizing set and an NR-basis for $\overline{H}_i$. Minor modifications of the above proof would yield ideals E and D of R with $\Lambda_i = \sum_{b \in B_i} Fb$ a ring of H_i, and $H_i D H_i \subset \Lambda_i$.

The next few results are concerned with a right T-module X which, when viewed as a right R-module, is torsion-free. The notation $\{b_1, \dots, b_v\}$, F and D of B.6 will be retained.

LEMMA B.7. *Let $Y \lhd X_R$ and set $Y_i = \{x \in XF \mid xb_i \subset Y\}$ and $\beta(Y) = \bigcap_1^v Y_i$. Then*

(i) *Y_i and $\beta(Y) \lhd XF_R$,*

(ii) *the map $x \mapsto xb_i$ from XF to X induces an embedding $\mathfrak{L}(XF/Y_i) \rightarrowtail \mathfrak{L}(X/Y)$ of lattices of R-submodules and*

(iii) *if Y is essential in X_R then Y_i and $\beta(Y)$ are essential in XF_R.*

PROOF. (i) is clear and (ii) is easily verified, since $b_i R = Rb_i$.

(iii) Suppose $0 \neq A \lhd XF_R$. If $Ab_i = 0$ then $A \subset Y_i$, while if $Ab_i \neq 0$ then $Ab_i \cap Y \neq 0$ and then $A \cap Y_i \neq 0$. Therefore Y_i is essential for each i, and then so is $\beta(Y)$. □

PROPOSITION B.8 (CF. [**16**, 24]). *If Y_R is essential in X_R then $Y \supset H \lhd X_T$ with H_R essential in X_R.*

PROOF. Since X_R is torsion-free, $X'E$ is essential in X'_R whenever $0 \neq E \lhd R$ and $X' \lhd X_R$. Now consider the chain

$$\beta(Y)D \subset \beta(Y)TDT \subset \beta(Y)\Lambda = \beta(Y)\left(\sum_1^v Fb_i\right) \subset Y.$$

Since $\beta(Y)D_R \subset \beta(Y)_R \subset Y_R$, with each essential in the next, choose

$$H = \beta(Y)(TDT). \quad \square$$

LEMMA B.9. (i) *X_R contains a submodule Y_R maximal with respect to $\beta(Y) = 0$.*

(ii) *For any $Y \lhd X_R$, $\beta(Y) \neq 0$ if and only if Y contains a nonzero T-submodule of X.*

PROOF. (i) This follows by a routine application of Zorn's lemma.

(ii) If $\beta(Y) \neq 0$ then $0 \neq \beta(Y)TDT \subset Y$, as in the previous proof. Conversely, if $0 \neq H_T \subset Y$ then $HFb_i \subset HT \subset Y$ for all i, so $0 \neq HF \subset \beta(Y)$. $\square$

PROPOSITION B.10. *Let $Y \lhd X_R$ be maximal with respect to $\beta(Y) = 0$, and suppose* u. dim.$(X_T) = m < \infty$. *Then*

(i) u. dim.$(X/Y)_R \leqslant m$ *and*

(ii) u. dim.$(X_R) \leqslant mv$ (*where* $v =$ u. dim. ${}_RT_R$).

PROOF. (i) Suppose $Y \underset{\neq}{\subset} A_i \lhd X_R$ with $\Sigma_1^t(A_i/Y)$ a direct sum in X/Y. For each i, $\beta(A_i) \neq 0$ and A_i contains $0 \neq (B_i)_T$, by B.9. If $t > m$, then

$$B = B_j \cap \left(\sum_{\substack{i=1\\ i\neq j}}^t B_i\right) \neq 0$$

for some j, and $B \subset A_j \cap (\Sigma_{i\neq j}A_i) = Y$, so $\beta(B) = 0$, contradicting B.9. Therefore $t \leqslant m$, proving (i).

(ii) Since $0 = \beta(Y) = \bigcap_1^v Y_i$, XF_R embeds in $\coprod_1^v(XF/Y_i)$. But, by B.7(ii), u. dim.$(XF/Y_i)_R \leqslant$ u. dim.$(X/Y)_R$ and so u. dim.$(XF_R) \leqslant mv$. But XF is essential in X_R (by the torsion-freeness of X_R), and so u. dim.$(X_R) \leqslant mv$. $\square$

COROLLARY B.11. (i) *If X_T is uniform then* u. dim.$(X_R) \leqslant$ u. dim.$({}_RT_R) \leqslant n$;

(ii) *If X_T is simple then X_R is semisimple with composition length at most n*;

(iii) *if $(T/J)_T$ has finite uniform dimension, so do $(T/J)_R$ and R_R*; *and*

(iv) *if T/J is prime right Goldie then R is prime right Goldie.*

PROOF. (i) This is simply B.10(ii) for $m = 1$.

(ii) Suppose X_T is simple. With notation as in B.10, if $0 \neq A/Y \lhd X/Y$ then $\beta(A) \neq 0$ and then (using B.9) $A = X$: therefore $(X/Y)_R$ is simple and so by B.7, each $(XF/Y_i)_R$ is simple or zero. Therefore XF is semisimple of finite length. But XF is essential in X_R, so it follows (from B.8) in this case that $XF = X$.

(iii) This follows immediately from B.10(ii), with $X_T = (T/J)_T$. To verify that $(T/J)_R$ is torsion-free, note that if $t \in T$ and $0 \neq D \lhd R$ with $tD \subset J$, then (since ${}_RT_R$ is torsion-free) $TD' \subset DT$ for some nonzero ideal D' of R, by [**6**, 2.2]. Then $tTD' \subset J$, with $D' \not\subset J$, and so $t \in J$.

(iv) This follows from (iii) and the fact that R inherits a.c.c. on right annihilators from T/J. $\square$

APPLICATION B.12. With notation as in 4.6, each X_i is semisimple with composition length at most n.

APPLICATION B.13. With notation as in 6.2 and 6.7,

(i) the uniform dimension of X_i, as an $f_i R$-module, is at most n;

(ii) any essential R_i submodule of X_i contains a nonzero T_i-submodule;

(iii) if T_i/J_i has finite right uniform dimension, then so does $f_i R$;

(iv) if T_i/J_i is prime right Goldie, then so too is $f_i R$.

The proof of the next result is an adaptation of that of [**16**, 3.2].

THEOREM B.14. *With Z the inverse image in T of the right singular ideal of T/J, and $Z(R)$ the (right) singular ideal of R, then $Z \cap R = Z(R)$.*

PROOF. ${}_R(M/J)_R$ is a normalizing (R, R)-bimodule generated by the cosets $\bar{x}_i = x_i + J$. Let Γ be a subset of $\{1,\ldots,n\}$ maximal with respect to the property that $\Sigma_{\gamma\in\Gamma} R\bar{x}_\gamma$ is a free left R-module with $\{\bar{x}_\gamma \mid \gamma \in \Gamma\}$ as basis. Then, as in [**16**, 2.1], it follows that

(i) ${}_R F = \Sigma_{\gamma\in\Gamma} R x_\gamma$ is free,

(ii) if $\{r_\gamma \mid \gamma \subset \Gamma\} \subset R$ with $\Sigma r_\gamma x_\gamma \in J$ then each $r_\gamma = 0$,

(iii) $DM \subset F + J$ for some $0 \neq D \lhd R$, and

(iv) $F \cap J = 0$.

Since ${}_R M_R$ is torsion-free, there exists $0 \neq D' \lhd R$ with $TD' \subset DT$.

Suppose first that $r \in Z(R)$, and suppose $0 \neq (K/J) \lhd_{\mathrm{rt}} T/J$. Then $KD'T \subset TD'T \subset DT \subset F + J$, but $KD'T \not\subset J$ (since $J \cap R = 0$ and $K \not\subset J$) so every element of $KD'T$ may be written $y = \Sigma_{\gamma\in\Gamma} r_\gamma x_\gamma + w$ with $r_\gamma \in R$, $w \in J$. Choosing an element y not in J for which, in such a representation, $\{\gamma \mid rr_\gamma = 0\}$ is as large as possible, it follows (for details, see [**16**]), that $y \in KD'T \sim J$ but $rr_\gamma = 0$ for *each* γ, so $ry \in J$. It now follows that $r \in Z$.

Conversely, suppose $r \in R \cap Z$ and let $0 \neq K \lhd_{\mathrm{rt}} R$. Then $KDT \subset KF + J$ but $KDT \not\subset J$ (otherwise $KD \subset J \cap R = 0$). Since $r \in Z$ there exists an element y in $KDT \sim J$ with $ry \in J$. Writing $y = \Sigma_\gamma k_\gamma x_\gamma + w$ $(k_\gamma \in K, w \in J)$, $r(y - w) \in F \cap J = 0$ so $rk_\gamma = 0$ for each γ. Since $y \notin J$, $k_\gamma \neq 0$ for some γ (by the definition of Γ) and so $K \cap \text{rt.ann.}_R(r) \neq 0$. This shows $r \in Z(R)$. $\square$

APPLICATION B.15. With the notation of 6.1, $Z_i \cap f_i R = Z(f_i R)$, the right singular ideal of $f_i R$.

REFERENCES

1. S. A. Amitsur, *Rings of quotients and Morita contexts*, J. Algebra **17** (1971), 273–298.

2. J. Bit-David and J. C. Robson, *Normalizing extensions*. I, Ring Theory 1980 (F. Van Ostaeyen, ed.), Lecture Notes in Math., vol. 825, Springer-Verlag, Berlin and New York, 1980.

3. J. Bit-David, *Normalizing extensions*. II, Ring Theory 1980 (F. Van Ostaeyen, ed.), Lecture Notes in Math., vol. 825, Springer-Verlag, Berlin and New York, 1980.

4. E. Formanek and A. V. Jategaonkar, *Subrings of Noetherian rings*, Proc. Amer. Math. Soc. **46** (1974), 181–186.

5. A. G. Heinicke and J. C. Robson, *Normalizing extensions: prime ideals and incomparability*, J. Algebra **72** (1981), 237–268.

6. ______, *Normalizing extensions: nilpotency*, J. Algebra **76** (1982), 459–470.
7. ______, *Normalizing extensions: prime ideals and incomparability.* II, J. Algebra (to appear).
8. S. Jabbour, *Intermediate normalizing extensions*, Comm. Algebra (to appear).
9. ______, *Chain conditions in intermediate normalizing extensions*, Comm. Algebra (to appear).
10. C . Lanski, *Goldie conditions in finite normalizing extensions*, Proc. Amer. Math. Soc. **79** (1980), 515–519.
11. M. Lorenz, *Finite normalizing extensions of rings*, Math. Z. **176** (1981), 447–484.
12. W. K. Nicholson and J. F. Watters, *Normal classes of rings and tensor products*, Comm. Algebra **9** (1981), 299–311.
13. D. S. Passman, *Prime ideals in normalizing extensions*, J. Algebra **73** (1981), 556–572.
14. R. Resco, *Radicals of finite normalizing extensions*, Comm. Algebra **7** (1981), 713–725.
15. J. C. Robson, *Prime ideals in intermediate extensions*, Proc. London Math. Soc. (3) **44** (1982), 372–384.
16. P. N. Stewart, *Properties of intermediate extensions*, Comm. Algebra (to appear).
17. P. N. Stewart and J. F. Watters, *Properties of normalizing extensions and fixed rings*, Comm. Algebra (to appear).
18. R. B. Warfield, Jr., *Prime ideals in ring extensions*, J. London Math. Soc (2) (to appear).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WESTERN ONTARIO, LONDON, ONTARIO, CANADA N6A 5B7

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF LEEDS, LEEDS, ENGLAND